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# INTEGRATIONS OF INVARIANT MEASURES OVER THE MODULI SPACE OF RIEMANN SURFACES

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ABSTRACT. This paper concerns with the convergence of the integral of invariant measures on the moduli space of Riemann surfaces.

## 1. INTRODUCTION

Let $X$ be a Riemann surface of analytically finite type $(g, n)$ with $2g-2+n>0$. Denote by $T(X)$ and $\mathcal{M}_{g,n}$ the Teichmüller space of $X$ and the moduli space of Riemann surface of the same type as $X$. It is known that $\mathcal{M}_{g,n}$ is the quotient space of $T(X)$ by the Teichmüller modular group $\mathrm{Mod}(g,n)$ (cf. §2.1.1).

Let $\mathcal{CM}$ be the set of all complex manifolds. For $M \in \mathcal{CM}$, we denote by $\mathrm{Meas}(M)$ the set of Borel measures on $M$. In this paper, by an *invariant measure*, we mean an assignment $\mu_{\cdot}\colon \mathcal{CM} \ni M \mapsto \mu_M \in \mathrm{Meas}(M)$ with the following properties.

(1) When $M$ is equal to the unit ball $\mathbb{B}^N$ in $\mathbb{C}^N$, $\mu_{\mathbb{B}^N}$ is the measure $\mu^{ber}_{\mathbb{B}^N}$ defined from the Bergman metric on $\mathbb{B}^N$ (cf. §3.4).
(2) For any $M_1, M_2 \in \mathcal{CM}$ and any holomorphic mapping $f\colon M_1 \to M_2$,
$$\mu_{M_2}(f(E)) \leq \mu_{M_1}(E)$$
for any Borel set $E \subset M_1$.

By definition, any invariant measure $\mu_{\cdot}$ defines a *biholomorphic invariant*. Namely, for any $M_1, M_2 \in \mathcal{CM}$ and a biholomorphic mapping $f : M_1 \to M_2$, we have $f_*\mu_{M_1} = \mu_2$, where $f_*\mu_{M_1}$ stands for the *push-forward* of $\mu_M$ which is defined by
$$(f_*\mu_{M_1})(E) = \mu_{M_1}(f^{-1}(E))$$
for all Borel set $E \subset M_2$. For the simplicity for $M \in \mathcal{CM}$, we also call the assigned measure $\mu_M$ an *invariant measure* on $M$.

The aim of this short paper is to give a criterion for the convergence of the integration by invariant measures on the moduli space as follows.

**Theorem 1** (Convergence of invariant measures). *Fix a point $x_0 \in \mathcal{M}_{g,n}$. Let $\phi : [0,\infty) \to [0,\infty)$ be a non-decreasing function. Let $\nu$ be a volume element on*

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*$T(X)$ which is defined from either an invariant measure or the Kälher-Einstein metric on $T(X)$. If $\phi(t) = O(e^{pt})$ with $0 \leq p < 2$ as $t \to \infty$, then*

$$\int_{\mathcal{M}_{g,n}} \phi(d_{\mathcal{M}}(x_0, x))d\nu(x) < \infty, \tag{1.1}$$

*Furthermore, the same conclusion holds if $\nu$ is the $(6g-6+2n)$-dimensional Hausdorff measure with respect to any invariant distance.*

Since measures on $T(X)$ given in Theorem 1 is invariant under the action of $\mathrm{Mod}(g,n)$, such measures descend as measures to $\mathcal{M}_{g,n}$. The integration (1.1) stands for the integration of the descending measure from $\nu$ over $\mathcal{M}_{g,n}$. Another description of the integration (1.1) is the integration by the push-forward measure

$$(\mathrm{proj}_*(\nu\,|_{\mathcal{M}_0}))(E) = \nu(\mathrm{proj}^{-1}(E) \cap \mathcal{M}_0)$$

on $\mathcal{M}_{g,n}$, where $E \subset \mathcal{M}_{g,n}$ is a Borel set, $\mathcal{M}_0 \subset T(X)$ is a fundamental domain of the action of $\mathrm{Mod}(g,n)$ such that $\partial\mathcal{M}_0$ is measure zero, and proj is the projection $T(X) \to \mathcal{M}_{g,n}$.

Here, $d_{\mathcal{M}}$ is defined by

$$d_{\mathcal{M}}(x,y) = \inf_{\omega \in \mathrm{Mod}(g,n)} d_T(x, \omega(y))$$

for $x, y \in T(X)$, where $d_T$ is the Teichmüller distance (cf. §2.1.1). It is easy to see that $d_{\mathcal{M}}$ is canonically recognized as a distance on the moduli space $\mathcal{M}_{g,n}$.

**Corollary 1** (Second moment is finite). *Let $\nu$ be a measure as Theorem 1.1. Then, the second moment*

$$\int_{\mathcal{M}_{g,n}} d_{\mathcal{M}}(x_0, x)^2 d\nu(x)$$

*is finite.*

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**Notation.** We use the notation $A \lesssim B$ to mean $A \leq CB$ for some constant $C > 0$ which is dependent on the topology of $X$ but it is otherwise universal.

## 2. Notation

### 2.1. Teichmüller theory.

In this section, we recall some fundamentals in Teichmüller theory. For details, see [7] and [8] for instance.

2.1.1. *Teichmüller space and Moduli space.* The *Teichmüller space* $T(X)$ of $X$ is the set of equivalence classes of marked Riemann surfaces $(Y, f)$ where $Y$ is a Riemann surface and $f: X \to Y$ a quasiconformal mapping. Two marked Riemann surfaces $(Y_1, f_1)$ and $(Y_2, f_2)$ are *Teichmüller equivalent* if there is a conformal mapping $h: Y_1 \to Y_2$ which is homotopic to $f_2 \circ f_1^{-1}$ (cf. [8]).

Teichmüller space $T(X)$ has a canonical complete distance, which is called the *Teichmüller distance* $d_T$ which is defined by

$$d_T(y_1, y_2) = \frac{1}{2}\log\inf\{K(h) \mid h \text{ is q.c. homotopic to } f_2 \circ f_1^{-1}\} \tag{2.1}$$

for $y_i = (Y_i, f_i) \in T(X)$ $(i = 1, 2)$ and $K(h)$ the maximal dilatation of $h$.

The *Teichmüller modular group* $\mathrm{Mod}(g,n)$ of $X$ is the factor group of the group of all quasiconformal self-mappings of $X$ over the normal subgroup of those homotopic to the identity (cf. [8]). Any element $\omega \in \mathrm{Mod}(g,n)$ acts $T(X)$ by

$$\omega(Y,f) = (Y, f\circ\omega^{-1}).$$

The action of each element of $\mathrm{Mod}(g,n)$ is holomorphic and $\mathrm{Mod}(g,n)$ acts on $T(X)$ properly discontinuously. The quotient space

$$\mathcal{M}_{g,n} = T(X)/\mathrm{Mod}(g,n)$$

is called the *Moduli space of Riemann surfaces of analytically finite type* $(g,n)$. The complex structure on $T(X)$ descends to the moduli space $\mathcal{M}_{g,n}$ and $\mathcal{M}_{g,n}$ has the structure of a normal complex space (cf. [8]).

2.1.2. *Teichmüller infinitesimal metric.* Let $L^\infty(Y)$ be the complex Banach space of bounded measurable $(-1,1)$-forms on a Riemann surface $Y$. Let $\mathcal{Q}_Y$ be the Banach space of integrable holomorphic quadratic differentials on $Y$ with norm

$$\|\varphi\| = \int_Y |\varphi(w)|dudv$$

where $w = u+iv$. The Teichmüller space $T(X)$ admits a canonical complex structure with the property that the holomorphic tangent space $T_yT(X)$ $(= T_y^{1,0}T(X))$ of $T(X)$ at $y \in T(X)$ is canonically identified with the quotient space

$$L^\infty(Y)/N(Y)$$

where

$$N(Y) = \{\mu \in L^\infty \mid \langle\varphi,\mu\rangle = 0, \text{ for all } \varphi \in \mathcal{Q}_Y\}$$

and

$$\langle\varphi,\mu\rangle = \int_Y \mu\varphi = \int_Y \mu(z)\varphi(z)dxdy$$

for $\mu \in L^\infty(Y)$ and $\varphi \in \mathcal{Q}_Y$ (cf. [8]). The *Teichmüller infinitesimal metric* is defined by

$$\kappa_T(y,[\mu]) = \sup\{|\mathrm{Re}\langle\varphi,\mu\rangle| \mid \|\varphi\| = 1, \varphi \in \mathcal{Q}_Y\}$$

where $y = (Y,f) \in T(X)$. The Teichmüller distance is characterized as the *inner distance* defined by $\kappa_T$. Namely, for $y_1, y_2 \in T(X)$,

$$d_T(y_1,y_2) = \inf_\gamma \int_0^1 \kappa_T(\gamma(t),\dot\gamma(t))dt \tag{2.2}$$

where $\gamma$ runs all $C^1$-paths $\gamma : [0,1] \to T(X)$ connecting $y_1$ and $y_2$. It is also known that $\kappa_T$ coincides with the Kobayashi-Royden intrinsic metric on Teichmüller space with respect to the canonical complex structure (See §3.2. See also [7]). Furthermore, H. Royden observed that $\kappa_T$ is continuous on the holomorphic tangent bundle over $T(X)$ (cf. [16]).

2.1.3. *Deligne-Mumford compactification.* Let $\overline{\mathcal{M}_{g,n}}$ be the Deligne-Mumford compactification of the moduli space $\mathcal{M}_{g,n}$. Let $\partial\mathcal{M}_{g,n} = \overline{\mathcal{M}_{g,n}} - \mathcal{M}_{g,n}$.

Any point of $\partial\mathcal{M}_{g,n}$ is a noded Riemann surface of type $(g,n)$ defined as follows. A *noded Riemann surface* is a one-dimensional complex space such that each point has either a neighborhood biholomorphic to a disc or to a neighborhood of $(0,0)$ in the variety defined by $xy=0$ in $\mathbb{C}^2$. A *node* is, by definition, is a point of the latter type. A noded Riemann surface is said to be *of type* $(g,n)$ if the *plumbing construction* (described below) at every node gives a Riemann surface of type $(g,n)$.

2.1.4. *Plumbing construction.* Let us start with a local description of the plumbing construction. Let $\mathbb{D}_z$ and $\mathbb{D}_w$ be unit disks in $z$ and $w$-planes respectively. Let $U_0$ be a union of $\mathbb{D}_z$ and $\mathbb{D}_w$ attached at their origins. Let $p_0 \in U_0$ be the point corresponding to the origins. This $U_0$ is a neighborhood of a node $p_0$. For $t \in \mathbb{D}$, let $\mathbb{D}_{z,t} = \mathbb{D}_z \setminus \{|z| \le |t|\}$ and $\mathbb{D}_{w,t} = \mathbb{D}_w \setminus \{|w| \le |t|\}$. Points $z \in \mathbb{D}_{z,t}$ and $w \in \mathbb{D}_{w,t}$ are said to be *equivalent* if $zw = t$. The resulting surface $U_t$ is said to be a surface constructed from the plumbing construction of parameter $t \in \mathbb{D}$.

In general, let $S$ be a noded Riemann surface and $p_1, \cdots, p_k$ nodes of $S$. Let $V_i$ be a neighborhood of $p_i$ which biholomorphic to $U_0$ above. Let $\mathbf{t} = (t_1, \cdots, t_k) \in \mathbb{D}^k$. We construct a noded Riemann surface $S_{\mathbf{t}}$ of type $(g,n)$ by modifying $V_i$ to $V_{i,t_i}$ with the plumbing construction of parameter $t_i$ for each $i$. We call $S_{\mathbf{t}}$ a *Riemann surface obtained by the plumbing construction with parameter* $\mathbf{t}$. By definition, $S_{\mathbf{t}}$ is a Riemann surface without nodes if and only if the product $t_1 \cdots t_k$ is not zero.

2.1.5. *Neighborhoods of boundary points.* A neighborhood of $S \in \partial\mathcal{M}_{g,n}$ is obtained as follows. Let $p_1, \cdots, p_{k_0}$ be nodes of $S$ and set $S' = S \setminus \{p_1, \cdots, p_{k_0}\} = S_1 \cup \cdots \cup S_{l_0}$. The Teichmüller space of $S'$ is the product space $\prod_i T(S_i)$ of the Teichmüller spaces of $S_i$.

For each $i$, we fix a neighborhood $V_i$ of $p_i$ homeomorphic to $U_0$ in §2.1.4. Let $V_i^1$ and $V_i^2$ be components of $V_i - p_i$ and fix a conformal mapping $z_i^a : V_i^a \to \mathbb{D}^* = \mathbb{D} \setminus \{0\}$ $(a = 1, 2)$. Choose $V_i$ sufficiently small with the following properties :

(i) $\overline{V_{i_1}} \cap \overline{V_{i_2}} = \emptyset$ for $i_1 \neq i_2$.
(ii) There is a system $\{\mu_j\}_{j=1}^{k_1}$ $(k_1 = \sum_{i=1}^{l} \dim_{\mathbb{C}} T(S_i))$ of Beltrami differentials on $S'$ such that
   (ii.1) each $\mu_j$ vanishes on $\cup_{i=1}^{k_0} V_i$ and
   (ii.2) linear combinations over $\mathbb{C}$ of $\{\mu_j\}_{j=1}^{k_1}$ spans the holomorphic tangent space of the product $\prod_i T(S_i)$ of the Teichmüller spaces of $S_i$ at $[S', id] = ([S_1, id], \cdots, [S_{l_0}, id])$.

Let $\mathbf{s} = (s_1, \cdots, s_{k_1}) \in \mathbb{D}_\epsilon^{k_1}$ where $\mathbb{D}_\epsilon = \{s \in \mathbb{C} \mid |s| < \epsilon\}$. Let $f_{\mathbf{s}}$ be the quasiconformal mapping on $S'$ with Beltrami differential $\sum_{j=1}^{N} s_j \mu_j$. We deform $S$ by $f_{\mathbf{s}}$ and apply the plumbing construction with parameter $\mathbf{t}$ under the local parameter $(f_{\mathbf{s}}(V_i^a), z_i^a \circ f_{\mathbf{s}}^{-1})$ at each node $f_{\mathbf{s}}(p_i)$. We denote by $S_{\mathbf{s},\mathbf{t}}$ the resulting Riemann surface under this operation. The following is known (cf. [8]).

**Proposition 2.1** (Bers). *There is a neighborhood $\mathcal{D}_S \subset \mathbb{D}^{k_0} \times \mathbb{D}_\epsilon^{k_1}$ of the origin and a finite group $G$ of biholomorphic automorphism acting on $\mathcal{D}_S$ such that*

$$\mathcal{D}_S/G \ni (\mathbf{t}, \mathbf{s}) \mapsto S_{\mathbf{t},\mathbf{s}} \in \overline{\mathcal{M}_{g,n}}$$

*is a holomorphic atlas around $S$.*

## 3. Hausdroff measure on the moduli space

3.1. **Hausdorff measures.** Let $X = (X, d)$ be a metric space. Let $\alpha > 0$. For $E \subset X$ and $\delta > 0$, we define

$$\mathcal{H}_\delta^\alpha(E; X) = \inf\left\{\sum_i \operatorname{diam}_X(U_i)^\alpha \mid E \subset \cup_i U_i,\ \operatorname{diam}_X(U_i) < \delta\right\}$$

where $\mathrm{diam}_X(E)$ is the diameter of $E \subset X$ in $(X, d)$. One can see that $\mathcal{H}^\alpha_{\delta_2}(E; X) \geq \mathcal{H}^\alpha_{\delta_1}(E; X)$ for $\delta_2 < \delta_1$ and hence the limit

$$\mathcal{H}^\alpha(E; X) = \lim_{\delta \to 0} \mathcal{H}^\alpha_\delta(E; X) = \sup_{\delta > 0} \mathcal{H}^\alpha_\delta(E; X)$$

is well-defined. We call $\mathcal{H}^\alpha(E; X)$ the $\alpha$-dimensional *Hausdorff measure* of $E$ on $(X, d)$. The following is well-known.

**Proposition 3.1** (Kolmogoroff principle (cf. (1.14) in [3])). *Let $f : (X_1, d_1) \to (X_2, d_2)$ be a locally L-Lipschitz map. Namely, there is a $\delta > 0$ such that for any $x_1, y_1 \in X_1$,*

$$d_2(f(x_1), f(y_1)) \leq L\, d_1(x_1, y_1)$$

*when $d_1(x_1, y_1) < \delta$. Then,*

$$\mathcal{H}^\alpha(f(E); X_2) \leq L^\alpha \mathcal{H}^\alpha(E; X_1)$$

*for any $\alpha > 0$ and $E \subset X_1$.*

3.2. **Hausdorff measures on Kobayashi hyperbolic manifolds.** Let $M$ be a complex manifold and $T_pM$ denotes the holomorphic tangent space of $M$ at $p \in M$. For $p \in M$ and $v \in T_pM$, we define the *Kobayashi-Royden infinitesimal metric* of $v$ at $p$ by

$$k_M(p; v) = \inf\{|\alpha| \mid \alpha \in T_0\mathbb{D} \text{ and } f : \mathbb{D} \to M \text{ hol. with } f(0) = p \text{ and } f_*(\alpha) = v\}.$$

The inner distance $d^{kob}_M$ defined from $k_M$ is called the *Kobayashi distance* on $M$ (cf. (2.2)). We say that $M$ is *Kobayashi hyperbolic* if $d^{kob}_M$ is a distance on $M$ (cf. [9]).

The *Kobayashi indicatrix* $I_K(p)$ at $p \in M$ is the unit ball with respect to $k_M$ in $T_pM$. Namely,

$$I_K(p) = \{v \in T_pM \mid k_M(p; v) < 1\}.$$

We denote by $\hat{I}_K(p)$ the convex hull of $I_K(p)$ in $T_pM$. Fix a holomorphic atlas $\mathbf{z} = (z_1, \cdots, z_m)$ around $p$ and $\varphi_p : T_pM \to \mathbb{C}^m$ the $\mathbb{C}$-linear trivialization induced from the atlas. We define

$$\Xi^K_M = \frac{1}{\mathbf{m}\left(\varphi_p\left(\hat{I}_K(p)\right)\right)} d\mathbf{m}(\mathbf{z}),$$

where

$$d\mathbf{m} = d\mathbf{m}_m = \left(\frac{\sqrt{-1}}{2}\right)^m dz_1 \wedge d\bar{z}_1 \cdots \wedge dz_m \wedge d\bar{z}_m$$

is the $2m$-dimensional Lebesgue measure. In [2], J. Bland and I. Graham obtained the following formula by applying Busemann's work in [3].

**Proposition 3.2** (Bland and Graham). *Let $M$ be a Kobayashi hyperbolic manifold. Suppose that $k_M$ is continuous on the holomorphic tangent bundle $TM$ of $M$. Then,*

$$\mathcal{H}^{2n}(E; (M, d^{kob}_M)) = \int_E \Xi^K_M \tag{3.1}$$

*for any Borel set $E$ in $M$.*

**Remark 3.1.** *In* [2], *Bland and Graham considered a volume form $\mathbf{m}(\mathbb{B}^n) \cdot \Xi^K_M$ instead of $\Xi^K_M$, where $\mathbb{B}^n$ is the unit ball in $\mathbb{C}^n$ with respect to the Euclidean metric on $\mathbb{C}^n$ (cf. Theorem 1 in* [2]*). However, the Hausdorff measure in the their paper is obtained from ours with being multiplied by $\mathbf{m}(\mathbb{B}^n)$ (cf. (1) in* [2]*). Hence, the formula* (3.1) *coincides with Bland-Graham's formula.*

3.3. **Hausdorff measures on poly-punctured disks.** Let $d_{\mathbb{D}^*}$ and $d_{\mathbb{D}}$ be the hyperbolic distances on $\mathbb{D}^*$ and $\mathbb{D}$ of curvature $-4$, respectively. It is known that the Poincaré metric densities $\lambda_{\mathbb{D}}$ and $\lambda_{\mathbb{D}^*}$ on $\mathbb{D}$ and $\mathbb{D}^*$ are

$$\lambda_{\mathbb{D}}(z) = \frac{1}{1-|z|^2}$$
$$\lambda_{\mathbb{D}^*}(z) = \frac{1}{2|z|\log\frac{1}{|z|}}.$$

Hence, for instance, the Poincaré distance on $\mathbb{D}^*$ satisfies

$$d_{\mathbb{D}^*}(z, x_0) = \frac{1}{2}|\log|\log|z||| + O(1) \tag{3.2}$$

for $0 < |z| < x_0 < 1$, where the $O(1)$-part in (3.2) depends only on $x_0$, since the length of the horocircle passing through $z$ is dependent only on $|z|$ and tends to zero if $|z| \to 0$.

Let $N \geq 1$ and $N_1, N_2 \geq 0$ with $N_1 + N_2 = N$. The Kobayashi infinitesimal metric $\lambda_{\mathbb{D}_{N_1,N_2}}$ and the Kobayashi distance $d_{\mathbb{D}_{N_1,N_2}}$ on $\mathbb{D}_{N_1,N_2} = (\mathbb{D}^*)^{N_1} \times \mathbb{D}^{N_2}$ are obtained by

$$\lambda_{\mathbb{D}_{N_1,N_2}}(\mathbf{z}; \mathbf{v}) = \max\{\lambda_{\mathbb{D}^*}(z^i)|v^i|, \lambda_{\mathbb{D}}(z^j)|v^j| \mid 1 \leq i \leq N_1,\ 1 \leq j - N_1 \leq N_2\}$$
$$d_{\mathbb{D}_{N_1,N_2}}(\mathbf{z}_1, \mathbf{z}_2) = \max\{d_{\mathbb{D}^*}(z_1^i, z_2^i), d_{\mathbb{D}}(z_1^j, z_2^j) \mid 1 \leq i \leq N_1,\ 1 \leq j - N_1 \leq N_2\}$$

(cf. [9]). Therefore, the Kobayashi indicatrix $I_K(z)$ of $\mathbb{D}_{N_1,N_2}$ at $z \in \mathbb{D}_{N_1,N_2}$ is

$$I_K(z) = \left\{ \mathbf{v} \in \mathbb{C}^N \mid |v^i| \leq \frac{1}{\lambda_{\mathbb{D}^*}(z^i)}, |v^j| \leq \frac{1}{\lambda_{\mathbb{D}}(z^j)},\ \begin{matrix} 1 \leq i \leq N_1 \\ 1 \leq j - N_1 \leq N_2 \end{matrix} \right\}.$$

Since $I_K(z)$ is convex, from Bland-Graham's formula (Proposition 3.2), we get the following.

**Proposition 3.3.** *For any Borel set $E \subset \mathbb{D}_{N_1,N_2}$, we have*

$$\begin{aligned}\mathcal{H}^{2N}(E; (\mathbb{D}_{N_1,N_2}, d^{kob}_{\mathbb{D}_{N_1,N_2}})) &= \frac{1}{\pi^N}\int_E \prod_{i=1}^{N_1} \lambda_{\mathbb{D}^*}(z_i)^2 \prod_{i=N_1+1}^{N} \lambda_{\mathbb{D}}(z_i)^2 d\mathbf{m}(\mathbf{z}) \\ &= \frac{1}{\pi^N}\int_E \prod_{i=1}^{N_1} \frac{1}{(2|z|\log\frac{1}{|z_i|})^2} \prod_{i=N_1+1}^{N} \frac{1}{(1-|z_j|^2)^2} d\mathbf{m}(\mathbf{z})\end{aligned}$$

*where* $\mathbf{m}$ *is the $2N$-dimensional Lebesgue measure.*

3.4. **Hausdorff measure on the unit ball.** Recall that the *Bergman metric* $ds^2_{ber}$ on the unit ball

$$\mathbb{B}^N = \{z \in \mathbb{C}^N \mid \|z\|^2 = |z^1|^2 + \cdots + |z^N|^2 < 1\}$$

is $ds^N_{ber} = \sum_{j,k} g_{j\bar{k}} dz^j d\bar{z}^k$ where

$$\begin{aligned} g_{j\bar{k}} &= -\frac{\partial^2}{\partial z^j \partial \bar{z}^k}\log(1-\|z\|^2)^{N+1} \\ &= \frac{N+1}{(1-\|z\|^2)^2}[(1-\|z\|^2)\delta_{jk} + z^k \bar{z}^j] \end{aligned}$$

(cf. e.g. Chapter 4 in [10]). Let $d^{ber}_{\mathbb{B}^N}$ the distance function defined by the Bergman metric and $\mu^{ber}_{\mathbb{B}^N}$ the volume element defined by the Bergman metric. Since $\mathbb{B}^N$ is homogeneous, the kobayashi distance $d^{kob}_{\mathbb{B}^N}$ and the Bergman distance $d^{ber}_{\mathbb{B}^N}$ are comparable. Hence, from Bland-Graham's formula, we have the following observation.

**Proposition 3.4.** *All three measures $\mathcal{H}^N(\cdot;(\mathbb{B}^N,d^{ber}_{\mathbb{B}^N}))$, $\mathcal{H}^N(\cdot;(\mathbb{B}^N,d^{kob}_{\mathbb{B}^N}))$, and $\mu^{ber}_{\mathbb{B}^N}(\cdot)$ are comparable. Namely, there is a constant $C_1,C_2>0$ such that*

$$C_1\mathcal{H}^N(E;(\mathbb{B}^N,d^{ber}_{\mathbb{B}^N}))\leq\mathcal{H}^N(E;(\mathbb{B}^N,d^{kob}_{\mathbb{B}^N}))\leq C_2\mathcal{H}^N(E;(\mathbb{B}^N,d^{ber}_{\mathbb{B}^N}))$$
$$C_1\mathcal{H}^N(E;(\mathbb{B}^N,d^{kob}_{\mathbb{B}^N}))\leq\mu^{ber}_{\mathbb{B}^N}(E)\leq C_2\mathcal{H}^N(E;(\mathbb{B}^N,d^{kob}_{\mathbb{B}^N}))$$

*for any Borel set $E\subset\mathbb{B}^N$.*

3.5. **Hausdorff measure on the moduli space.** In [4], S. Dowdall, M. Duchin and H. Masur observed the following (see also [12]).

**Proposition 3.5** (cf. Corollary 17 in [4]). *The volume of $\mathcal{M}_{g,n}$ in terms of the $6g-6+2n$-dimensional Hausdorff measure with respect to the Teichmüller distance is finite.*

Indeed, in [4], Dowdall, Duchin and Masur showed that the $(6g-6+2n)$-dimensional Hausdorff measure is comparable with the push-forward of the Veech-Masur measure on the holomorphic unit cotangent bundle. Hence, Proposition 3.5 follows from the finiteness of the volume of the unit cotangent bundle (cf. [11] and [17]). Notice that from Theorem 2 below, we can also see directly the finiteness of the volume of $\mathcal{M}_{g,n}$ by the $(6g-6+2n)$-dimensional Hausdorff measure.

We define
$$\mathcal{H}_{T(X)}(E)=\mathcal{H}^{6g-6+2n}(E;(T(X),d_T))$$
for a Borel set $E\subset T(X)$ and $m\geq 0$, where $\mathcal{H}^{6g-6+2n}(\,\cdot\,;T(X))$ is the $(6g-6+2n)$-dimensional Hausdorff measure on $T(X)$ in terms of the Teichmüller distance on $T(X)$. It is easy to see that $\mathcal{H}^{6g-6+2n}$ is equivariant under $\mathrm{Mod}(g,n)$ :
$$\mathcal{H}_{T(X)}(\omega(E))=\mathcal{H}_{T(X)}(E)$$
for all Borel set $E$ of $T(X)$ and $\omega\in\mathrm{Mod}(g,n)$ since $\mathrm{Mod}(g,n)$ acts on $T(X)$ isometrically. Hence, $\mathcal{H}_{T(X)}$ descends to $\mathcal{M}_{g,n}$. As discussed in Introduction, the integration over $\mathcal{M}_{g,n}$ by this descending measure is nothing but that in (1.1) for $\nu=\mathcal{H}_{T(X)}$.

**Theorem 2** (Convergence of integral for Hausdorff measure). *Let $\phi:[0,\infty)\to[0,\infty)$ be a function satisfying $\phi(t)=O(e^{pt})$ with $0\leq p<2$ as $t\to\infty$. we have*
$$\int_{\mathcal{M}_{g,n}}\phi(d_{\mathcal{M}}(x_0,x))d\mathcal{H}_{T(X)}(x)<\infty$$
*where $x_0\in\mathcal{M}_{g,n}$ is a fixed base point.*

*Proof.* Fix $x_0\in\mathcal{M}_{g,n}$. Since the boundary divisor $\partial\mathcal{M}_{g,n}$ is compact, it suffices to show that any $S\in\partial\mathcal{M}_{g,n}$ admits a neighborhood $U_S$ in $\overline{\mathcal{M}_{g,n}}$ such that
$$\int_{U_S\cap\mathcal{M}_{g,n}}\phi(d_{\mathcal{M}}(x_0,x))d\mathcal{H}_{T(X)}(x)<\infty$$
for any $p\geq 1$.

Let $k_0\geq 1$ be the number of nodes of $S$ and set $k_1=3g-3+n-k_0$. From Bers' theorem (Proposition 2.1), there is a neighborhood $\mathcal{D}_S\subset\mathbb{D}^{k_0}\times\mathbb{D}^{k_1}$ of the origin $\mathbf{0}=(0,0)\in\mathbb{D}^{k_0}\times\mathbb{D}^{k_1}$ and a holomorphic mapping
$$\Phi_S:\mathcal{D}_S\to\overline{\mathcal{M}_{g,n}}$$

such that $\Phi_S(\mathbf{0}) = S$ and $\Phi_S(\mathbf{t},\mathbf{s}) \in \partial\mathcal{M}_{g,n}$ if and only if some coordinate of $\mathbf{t}$ is zero. Furthermore, the image $\Phi_S(\mathcal{D}_S)$ covers a neighborhood of $S$. Take $\epsilon > 0$ such that $\mathbb{D}_\epsilon^{k_0} \times \mathbb{D}_\epsilon^{k_1} \subset \mathcal{D}_S$, where $\mathbb{D}_\epsilon = \{|z| < \epsilon\}$. From the above argument, we deduce

$$\Phi_S((\mathbb{D}_\epsilon^*)^{k_0} \times \mathbb{D}_\epsilon^{k_1}) = \mathcal{M}_{g,n} \cap \Phi_S(\mathbb{D}_\epsilon^{k_0} \times \mathbb{D}_\epsilon^{k_1}),$$

where $\mathbb{D}_\epsilon^* = \{0 < |z| < \epsilon\}$. For the simplicity, we set $\mathcal{D}_0 = (\mathbb{D}_{\epsilon/2}^*)^{k_0} \times \mathbb{D}_{\epsilon/2}^{k_1}$, $\mathcal{D}_1 = (\mathbb{D}_\epsilon^*)^{k_0} \times \mathbb{D}_\epsilon^{k_1}$. Take a neighborhood $U_S$ of $S$ in $\mathcal{M}_{g,n}$ with $U_S \subset \Phi_S(\mathcal{D}_0)$.

Fix $(\mathbf{t}_1,\mathbf{s}_1) \in (\mathbb{D}_\epsilon^*)^{k_0} \times \mathbb{D}_\epsilon^{k_1}$ and let $x_1 = \Phi_S(\mathbf{t}_1,\mathbf{s}_1)$. Since the Kobayashi distance has the distance decreasing property with respect to holomorphic mappings, $\Phi_S$ is 1-Lipschitz (cf. [9]). Therefore,

$$d_T(\Phi_S(\mathbf{t}_1,\mathbf{s}_1),\Phi_S(\mathbf{t}_2,\mathbf{s}_2)) \leq d_{\mathcal{D}_1}((\mathbf{t}_1,\mathbf{s}_1),(\mathbf{t}_2,\mathbf{s}_2))$$
$$\mathcal{H}_{T(X)}(E) \leq \mathcal{H}^{6g-6+2n}(\Phi_S^{-1}(E);(\mathcal{D}_1,d_{\mathcal{D}_1}^{kob}))$$

for $(\mathbf{t}_1,\mathbf{s}_1),(\mathbf{t}_2,\mathbf{s}_2) \in \mathcal{D}_0$ and any measurable set $E \subset \mathcal{M}_{g,n}$ by Kolmogoroff principle (Proposition 3.1).

Since

$$\exp\left(p\left\{\frac{1}{2}\log|\log|x|| + O(1)\right\}\right) \lesssim |\log|x||^{p/2}$$

as $x \to 0$, by Proposition 3.3 and (3.2), we have

$$\begin{aligned}
&\int_{U_S\cap\mathcal{M}_{g,n}} \phi(d_{\mathcal{M}}(x_0,x))d\mathcal{H}_{T(X)}(x)\\
&\lesssim \int_{U_S\cap\mathcal{M}_{g,n}} \exp\left(p\cdot d_{\mathcal{M}}(x_0,x)\right) d\mathcal{H}_{T(X)}(x)\\
&\leq \int_{U_S\cap\mathcal{M}_{g,n}} \exp\left(p(d_{\mathcal{M}}(x_0,x_1) + d_{\mathcal{M}}(x_1,x))\right) d\mathcal{H}_{T(X)}(x)\\
&\lesssim \int_{\mathcal{D}_0} \exp\left(p(d_{\mathcal{D}_1}((\mathbf{t}_1,\mathbf{s}_1),(\mathbf{t},\mathbf{s})))\right) d\mathcal{H}^{6g-6+2n}((\mathbf{t},\mathbf{s});\mathcal{D}_1)\\
&\lesssim \int_{0<r_1<\epsilon/2} \cdots \int_{0<r_{k_0}<\epsilon/2} \frac{\max_{1\leq i\leq k_0}\{|\log r_i|^{p/2}\}}{\prod_{i=1}^{k_0} r_i|\log r_i|^2} dr_1\cdots dr_{k_0}\\
&\lesssim \int_{0<r_1<\epsilon/2} \cdots \int_{0<r_{k_0}<\epsilon/2} \frac{dr_1\cdots dr_{k_0}}{\prod_{i=1}^{k_0} r_i|\log r_i|^{2-(p/2)}} < \infty
\end{aligned}$$

for all $p$ with $0 \leq p < 2$, where $\mathbf{t} = (t_1,\cdots,t_{k_0})$, $\mathbf{s} = (s_1,\cdots,s_{k_1}) \in \mathcal{D}_1$ and $r_i = |t_i|$ $(1 \leq i \leq k_0)$. □

## 4. Invariant distances and measures on complex manifolds

In this section, we recall *invariant distances* and *invariant volumes* on a family of complex manifolds. Recall that we denote by $\mathcal{CM}$ the set of all complex manifolds. For $M \in \mathcal{CM}$, we denote by $\mathrm{Dist}(M)$ the set of pseudo-distances on $M$.

### 4.1. Invariant distances.

An *invariant distance* $\mathfrak{d}_\cdot$ is an assignment $\mathcal{CM} \ni M \mapsto \mathfrak{d}_M \in \mathrm{Dist}(M)$ with the following properties.

(1) $\mathfrak{d}_{\mathbb{D}}$ is the Poincaré metric $\rho_{\mathbb{D}}$ on $\mathbb{D}$ with curvature $-4$, and
(2) Let $M_1, M_2 \in \mathcal{CM}$. For any holomorphic mapping $f\colon M_1 \to M_2$, we have

$$\mathfrak{d}_{M_2}(f(x),f(y)) \leq \mathfrak{d}_{M_1}(x,y)$$

for $x,y \in M_1$.

A typical example of invariant distances is to assign the Kobayashi distances to complex manifolds (cf. §3.2). There is another typical example called the *Carathéodory distance* which defined by

$$d_M^{car}(x,y) = \sup\{\rho_{\mathbb{D}}(x,y) \mid f\colon M \to \mathbb{D} \text{ holomorphic}\}$$

for $M \in \mathcal{CM}$ and $x, y \in M$. The following is well-known.

**Proposition 4.1.** *Let $\mathcal{CM} \ni M \mapsto \mathfrak{d}_M$ be an invariant distance. For any $M \in \mathcal{CM}$, we have*

$$d_M^{car}(x,y) \le \mathfrak{d}_M(x,y) \le d_M^{kob}(x,y)$$

*for $x, y \in M$.*

### 4.2. Invariant measures revisited.

4.2.1. *Carathéodory and Kobayashi-Eiesenman volume forms.* Let $D$ be a domain in $\mathbb{C}^m$. By the *Carathéodory volume form* $\mathrm{CV}_D$, we mean a volume from on $D$

$$\mathrm{CV}_D = \mathrm{CV}(\mathbf{z})d\mathbf{m}(\mathbf{z})$$

where

$$\mathrm{CV}(\mathbf{z}) = \sup\left\{|\det f'(\mathbf{z})| \mid f : D \to \mathbb{B}^m \text{ hol. with } f(\mathbf{z}) = 0\right\}.$$

The *Kobayashi-Eiesenman volume form* $\mathrm{KV}_D$ is defined by

$$\mathrm{KV}_D = \mathrm{KV}(\mathbf{z})d\mathbf{m}(\mathbf{z})$$

where

$$\mathrm{KV}(\mathbf{z}) = \inf\left\{\frac{1}{|\det f'(0)|} \mid f : \mathbb{B}^m \to D \text{ hol. with } f(0) = \mathbf{z}\right\}.$$

For any $M \in \mathcal{CM}$, these volume forms $\mathrm{CV}_M$ and $\mathrm{KV}_M$ are defined in the same way. The following is well-known.

**Proposition 4.2.** *Let $M \in \mathcal{CM}$ and $d\mu$ a volume form on $M$. Let $C > 0$*

(1) *Suppose that for every holomorphic mapping $f\colon \mathbb{B}^N \to M$, $\mu$ satisfies*

$$C \cdot f^* d\mu \le d\mu_{\mathbb{B}^N}^{ber}$$

*on $\mathbb{B}^N$. Then, $C \cdot d\mu \le \mathrm{KV}_M$.*

(2) *Suppose that for every holomorphic mapping $f\colon M \to \mathbb{B}^N$, $\mu$ satisfies*

$$f^* d\mu_{\mathbb{B}^N}^{ber} \le C \cdot d\mu$$

*on $M$ Then, $\mathrm{CV}_M \le C \cdot d\mu$.*

4.2.2. *Carathéodory and Kobayashi invariant measures.* There are two typical examples for invariant measures (cf. IX.1 in [9]). The first is called the *Kobayashi intrinsic volume* defined as follows. Let $M \in \mathcal{CM}$ with $m = \dim_{\mathbb{C}}(M)$. For a Borel set $E \subset M$, we choose holomorphic mappings $f_i : \mathbb{B}^m \to M$ and Borel sets $E_i \subset \mathbb{B}^m$ with

$$E \subset \cup_i f_i(E_i). \tag{4.1}$$

Then, the measure $\mu_M^{kob}$ is defined by

$$\mu_M^{kob}(E) = \inf_{f_i, E_i} \sum_i \mu_{\mathbb{B}^N}^{ber}(E_i),$$

where $f_i$ and $E_i$ runs over all holomorphic mappings and Borel sets satisfying (4.1). We call $\mu_M^{kob}$ is the *Kobayashi intrinsic volume* on $M$. The second example is called the *Carathéodory intrinsic volume* which defined by

$$\mu_M^{car}(E) = \sup_f \mu_{\mathbb{B}^N}^{ber}(f(E))$$

for $M \in \mathcal{CM}$ and a Borel set $E \subset M$, where $f$ runs over all holomorphic mappings $M \to \mathbb{B}^m$. The following are well-known (cf. Chapter IX of [9], [6] and §2.4 of [14]).

**Proposition 4.3.** *For any invariant measure $\mu_\cdot$,*

$$\mu_M^{car}(E) \leq \mu_M(E) \leq \mu_M^{kob}(E)$$

*for any $M \in \mathcal{CM}$ and a Borel set $E \subset M$.*

**Proposition 4.4.** *For any $M \in \mathcal{CM}$ and a Borel set $E \subset M$,*

$$\mu_M^{kob}(E) = \int_E \mathrm{KV}(E).$$

4.3. **The case of Teichmüller spaces.** E. Overholser observed the following (cf. Lemma 3.1 in [15]).

**Proposition 4.5** (Overholser). *The following hold.*

(1) $\mathrm{CV}_{T(X)}$ *and* $\mathrm{KV}_{T(X)}$ *are comparable.*
(2) *Suppose that $D$ is a domain of holomorphy. Let $g$ be the Kähler-Einstein metric on $D$ and* $\mathrm{Vol}_g$ *the volume form defined from $g$. Then*

$$\mathrm{CV}_D \leq \mathrm{Vol}_g \leq \mathrm{KV}_D$$

*on $D$.*

4.4. **Proof of Theorem 1.** We are now ready to prove Theorem 1. In showing the former claim, for the simplicity, we only treat the case where $\nu$ is defined by an invariant measure. The remaining case is treated in the same way.

We set $N = 6g-6+2n$. Let $f : \mathbb{B}^N \to T(X)$ and $g : T(X) \to \mathbb{B}^N$ be holomorphic mappings. From Kolmogoroff principle (Proposition 3.1) and Proposition 3.4, we have

$$C_1\mathcal{H}_{T(X)}(f(E)) \leq C_1\mathcal{H}^N(E;(\mathbb{B}^N, d_{\mathbb{B}^N}^{ber})) \leq \mu_{\mathbb{B}^N}^{ber}(E)$$
$$\mu_{\mathbb{B}^N}^{ber}(g(E')) \leq C_2\mathcal{H}^N(g(E');(\mathbb{B}^N, d_{\mathbb{B}^N}^{ber})) \leq C_2\mathcal{H}_{T(X)}(E')$$

for Borel sets $E \subset \mathbb{B}^N$ and $E' \subset T(X)$. Therefore, their infinitesimal froms satisfy

$$f^*\Xi_{T(X)}^K \lesssim d\mu_{\mathbb{B}^N}^{ber}, \quad \text{and} \quad g^*d\mu_{\mathbb{B}^N}^{ber} \lesssim \Xi_{T(X)}^K.$$

Hence, by Propositions 4.2 and 4.5,

$$\mathcal{H}_{T(X)}(E) \lesssim \int_E \mathrm{KV}_{T(X)} \lesssim \int_E \mathrm{CV}_{T(X)} \lesssim \mathcal{H}_{T(X)}(E)$$

for any Borel set $E \subset T(X)$. Therefore, by Propositions 4.5 and 4.4, $\mu_{T(X)}^{kob}$ is comparable with $\mathcal{H}_{T(X)}$ and hence $\nu \lesssim \mathcal{H}_{T(X)}$. Thus, we conclude he convergence of the integral (1.1) with respect to $\nu$ from Theorem 2.

For the latter claim, let $\mathfrak{d}_\cdot : \mathcal{CM} \ni M \to \mathfrak{d}_M \in \mathrm{Dist}(M)$ be an invariant distance. Then, Kolmogoroff principle and Proposition 4.1 imply that

$$\mathcal{H}^{6g-6+2n}(E;(T(X),\mathfrak{d}_{T(X)})) \leq \mathcal{H}^{6g-6+2n}(E;(T(X),d_{T(X)}^{kob})) = \mathcal{H}_{T(X)}(E) \tag{4.2}$$

for any Borel set $E \subset T(X)$. Thus, from Theorem 2, we have what we wanted. $\square$

**Remark 4.1.** *The first and last term in* (4.2) *are comparable. Indeed, in* [5], *C. Earle observed that the Carathéodory distance is locally comparable with the Kobayashi distance on* $T(X)$ *(see also* [13]*). Hence, Kolmogoroff principle also asserts that* $\mathcal{H}^{6g-6+2n}(\,\cdot\,;(T(X),\mathfrak{d}_{T(X)}))$ *is comparable with* $\mathcal{H}_{T(X)}(\,\cdot\,)$.

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